Transients in Mechanical Systems

By J. T. MULLER

Introduction

A study of the response of an electrical network or system to the input of transients in the form of short-duration pulses is an accepted method of analysis of the network. By comparing the input and the output, conclusions may be drawn as to the respective merit of the various components.

Until recently similar procedures were only of academic interest with mechanical systems. However, tbe tests for mechanical ruggedness, which are required of electronic gear in order to pass specifications for the armed forces, are an example of the application of transients to a mechanical system. These tests are known as *High Impact Shock Tests*.

A basic part of an electrical system is a damped resonant network consisting of an inductance, a capacitance and a resistance. A mass, a spring and a friction device is the equivalent mechanical network called a simple mechanical system and a combination of such networks is a general mechanical system. It is, of course, advantageous to keep the mechanical system as simple as possible without detracting from the general usefulness of the results obtained.

The problems here considered are pertinent to a system which is essentially made up of a supporting structure or table and a resilient mounting array bearing the equipment (e.g. electronic gear) wbich is vulnerable to shock. (See Fig. 1.)

A shock is the physical manifestation of the transfer of mechanical energy from one body to another during an extremely short interval of time. The order of magnitude of the time interval is milliseconds and quite frequently fractions of a millisecond.

The system is excited by administering large spurts of mechanical energy to the supporting table. The manner in which this energy is supplied to the base and the way it is dissipated through the system are the subjects of this paper.

The energy transfer to the supporting table is accomplished by the use of huge hammers which strike the anvil with controllable speeds. The action is assumed to be similar to that of an explosion, particularly to an underwater explosion at close range or a *near-miss*. As to the real comparison between the two, the reader is referred to the various manuscripts published by the Bureau of Ships. This particular phase of the subject is

considered outside the scope of this paper, except for the following brief statement:

Both actions fit the definition of shock stated above and the difference between the two is one of size and not of kind.

Shocks are transients and are conveniently treated by a branch of mathematics which is adapted to the solution of problems of this kind; viz, the *Laplace Transforms*, and the reader is referred to Gardner and Barnes, "Transients in Linear Systems." The nomenclature used here is identical to that of those authors.

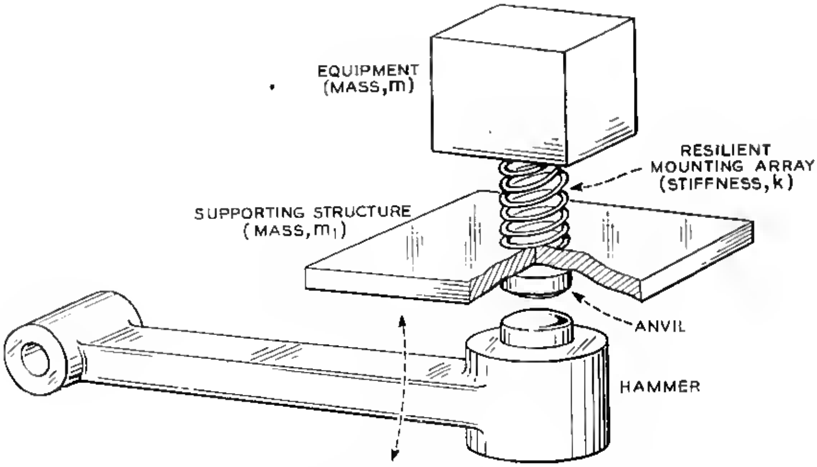

Fig. 1—Schematic layout of shock machine.

The manuscript consists of two parts: In the first, the energy transfer to the base is considered. We are dealing here with rigid bodies; consequently with very small transient displacements and very large forces. These are usually referred to as impact forces or impulses and four such functions of force and time are discussed. Displacements with associated velocities result from the action of impulses on the base.

The second part deals with the effect of these displacements on the shock-mounted equipment. Although the mathematical procedure is identical to the first part, here we deal with a function of displacement and time. There is no specific name for such a relationship but a suggestive term is "whip." However, the pulse functions represented are the same as those of the force and time function.

It is assumed that the displacement-time pulse is independent of the subsequent motion of the mass.

In considering any kind of shock problem we have the following fundamental considerations:

First, we shall want to know the magnitude of the shock present in the base or supporting structure; this will be called the "excitation."

Second, the behavior of the resilient medium interposed between the shock-producing base and the equipment. It is sometimes expressed as the coupling. We shall use the term *transmission.*

Third, the resulting disturbance of the equipment caused by the transmitted shock, which we will call the *response.*

The three functions do not exist independently, but are mathematically related. For a clearer understanding of shock phenomena it is perhaps helpful to fix in one's mind the idea that the response of a system is completely dependent upon the transmission function.

To use an electrical analogue, the voltage e_1 (t) impressed upon a system produces an output voltage e_2 (t) which is completely defined by the transmission function. For instance, if this transmission function represents a filter of some kind with given boundaries, then it is to be expected that the response of e_2 (t) is completely changed outside these limits and could even be zero. The same train of thought will hold for mechanical systems. Here the transmission function is mostly represented by the stiffness or the compliance. For a completely rigid medium the stiffness would be infinite and the input and output would be alike; in other words, a force applied to the base would appear at the equipment. This is a theoretical case because no material is perfectly rigid. Though some materials are more rigid than others they will all give if the force applied is big enough. Now the forces associated with a shock are almost always of considerable magnitude so that the stiffness of a material becomes significant.

As the stiffness diminishes the response changes and may appear to be quite different from the input. As far as the transmissibility of forces is concerned, the reader is reminded that a force is always accompanied by a reaction. The forces which put the base into motion cannot be transmitted by a soft material like rubber, unless it is compressed to extremely high values, and thus produce an equally large reactive force.

PART I

Analysis of The Excitation of the Base

By recording the motions of the base, we obtain time-displacement curves as shown in Fig. 2. The method of recording has been done by means of high-speed motion pictures (at the Whippany testing laboratory using a Fastex) and by using strain gages (at the Annapolis Engineering Experiment Station).

The test equipments are fundamentally mechanical impact producing machines. For technical details and description of the machines the reader

is referred to the various test specifications by the Bureau of Ships (as, for example, *Spec.* 40T9).

The characteristic of an impact is the transfer of mechanical energy from one mass to another in a relatively short time. The corresponding force as function of time is called an *impulse*, henceforth indicated as $F(t)$. A study of the pulse functions has suggested some prohahle theoretical shapes of $F(t)$ which could cover a wide variety of conditions. These pulse functions will he used for force-time functions as well as displacement-time-functions and it will be shown that the results are surprisingly similar.

We will let these pulses operate on the hase with mass m_1 and calculate and plot the resulting time displacement curves. Since an impulse is associated with energy transfer, it must be a function of $\frac{m_1 v^2}{2}$. From the point of view

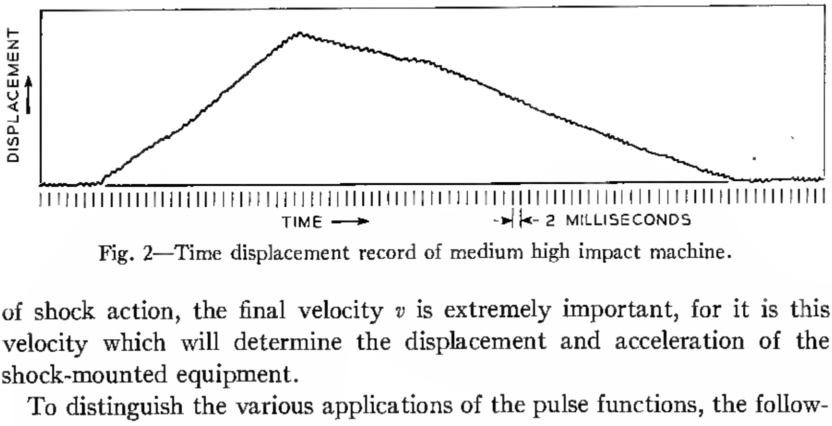

Fig. 2—Time displacement record of medium high impact machine.

of shock action, the final velocity v is extremely important, for it is this velocity which will determine the displacement and acceleration of the shock-mounted equipment.

To distinguish the various applications of the pulse functions, the following notations are adopted:

$f(t)$ represents any functions of t, without reference to its dimensional magnitude. The transform of $f(t)$ is indicated by $F(s)$.

$x(t)$ represents a function of t when it is a displacement of the mass m only. The transform is indicated by $X(s)$.

$x_1(t)$ represents a function of t when it is a displacement of the *base* (with mass m_1) only. The transform is indicated hy $X_1(s)$.

$F(t)$ represents a function of t when it is a force applied to the base. The transform is indicated by $F_0(s)$.

Since $x_1(t)$ and $F(t)$ are input functions, they may he represented by the same type pulse, in which case the transforms are alike, i.e., $F(s) = X_1(s) = F_0(s)$.

Figure 3A, a rectangular pulse, is the simplest form.

Figure 3B is a triangular pulse, $f(t)$, reaching a peak and returning to zero in a linear manner.

Figure 3C consists of one-half cycle of a sine wave.

Figure 3D is a cosine pulse of one-cycle duration and is shifted along the Y axis an amount equal to the amplitude.

These are the pulses to be used in the problems under consideration.

If they represent a force as it varies with time then it is said that $F(t)$ represents a particular pulse. The Laplace transform of $F(t)$ is given as $F_0(s)$, $F_0(s)$ being some function in the complex domain. It is outside the scope of this paper to prove or show the mathematical technique in obtaining the transforms which produce $F_0(s)$. We will present them here for future reference.

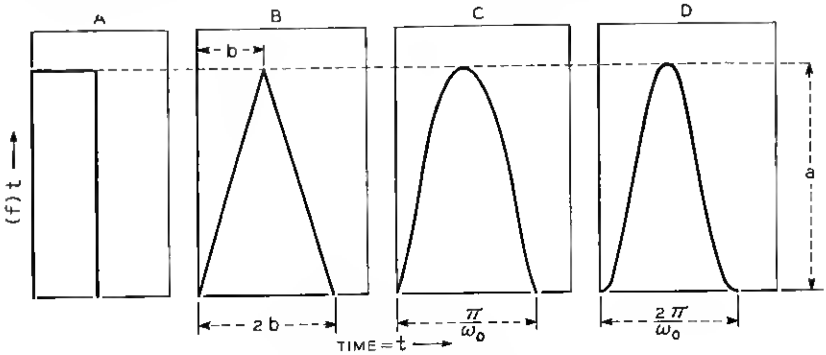

Fig. 3—Four pulses.

The Laplace transform for a very short pulse is

$$F(s) = A_r \tag{1.01}$$

and is a pulse which has a finite area but the time interval of which is approaching zero.

For a square pulse with finite time interval and magnitude a (hereafter referred to as pulse amplitude) it is

$$F(s) = a\,\frac{1 - \epsilon^{-bs}}{s} \tag{1.02}$$

For a triangular pulse

$$F(s) = \frac{a}{b}\left(\frac{1 - \epsilon^{-bs}}{s}\right)^2 \tag{1.03}$$

For a sine pulse

$$F(s) = \frac{a\omega_0}{s^2 + \omega_0^2}\,(1 + \epsilon^{-\pi s/\omega_0}) \tag{1.04}$$

For a shifted cosine pulse

$$F(s) = \frac{a\omega^2}{2s(s^2 + \omega_0^2)} (1 - \epsilon^{-2\pi s/\omega_0}) \tag{1.05}$$

Suppose we let an impulse, and to take a specific example, a triangular impulse, operate on a mass m_1. We have

$$F = m_1 a_0 \tag{1.06}$$

in which

$$F = F(t) = \text{force in lbs.}$$

$$m_1 = \text{mass in slugs}$$

$$a_0 = \ddot{x}_1 = \text{acceleration in ft/sec}^2$$

Let $\mathfrak{L}[x_1(t)] = X_1(s) = X_1$
then

$$\mathfrak{L}[m_1\ddot{x}_1] = m_1 s^2 X_1(s)$$

the $\mathfrak{L}$ transform of a triangular pulse $F(t)$ is

$$\mathfrak{L}[F(t)] = \frac{a}{b}\left(\frac{1 - \epsilon^{-bs}}{s}\right)^2 = F_0(s) \tag{1.07}$$

Substituting

$$X_1 = \frac{a}{m_1 b}\frac{1}{s^2}\left(\frac{1 - \epsilon^{-bs}}{s}\right)^2$$

The inverse transform is

$$\mathfrak{L}^{-1}[X_1(s)] = x_1(t) = \mathfrak{L}^{-1}\left[\frac{a}{m_1 b}\frac{1}{s^2}\left(\frac{1 - \epsilon^{-bs}}{s}\right)^2\right] \tag{1.08}$$

$$x_1(t) = \frac{a}{m_1 b}\mathfrak{L}^{-1}\left[\frac{1}{s^2}\left(\frac{1 - \epsilon^{-bs}}{s}\right)^2\right] \tag{1.09}$$

The solution of 1.09

$$x_1(t) = \frac{a}{m_1}\frac{1}{b}\left[\frac{t^3}{6} - 2\frac{(t-b)^3}{6}u(t-b) + \frac{(t-2b)^3}{6}u(t-2b)\right]. \tag{1.10}$$

After the impulse is over, i.e., for values of $t > 2b$, 1.10 becomes

$$x_1(t) = \frac{a}{m_1}b(t - b) \tag{1.11}$$

and the final velocity is

$$\frac{a}{m_1} b. \tag{1.12}$$

which represents the area of the impulse divided by the mass.

Similarly we find for the very short square impulse

$$x_1 = \frac{A_r}{m_1} t \tag{1.13}$$

For the square impulse of finite time interval b

$$x_1 = \frac{ab}{m_1}\left(t - \frac{b}{2}\right) \tag{1.14}$$

For the sine impulse

$$x_1 = \frac{2a}{m_1\omega_0}\left(t - \frac{\pi}{2\omega_0}\right) \tag{1.15}$$

For the shifted cosine impulse

$$x_1 = \frac{a\pi}{m_1\omega_0}\left(t - \frac{\pi}{\omega_0}\right) \tag{1.16}$$

The velocity is the term preceding the term in parenthesis.

In the five examples mentioned, we find that this *velocity is proportional to the area of the impulse curve and inversely proportional to the mass.* All expressions contain the factor $\frac{a}{m_1}$ and, since a is the maximum force present, this expression represents the maximum acceleration and it is this value which is so frequently mentioned when discussing the actions on the shock table.

For instance, from records we have determined approximate values for the time interval during which the energy transfer from hammer to the table takes place. The high-speed motion pictures are taken at the rate of 4,000 to 5,000 frames per second, which means an average elapsed time of .22 milliseconds or 220 microseconds. The energy transfer occurs within this time interval, because the rate of increase of the displacement from frame to frame is constant. The exposure time of one frame is $\frac{1}{12000}$ second or 83 microseconds. If the anvil moved within this time there would be evidence of blurring. Since we have been unable to detect any blurring, we may state that transfer is less than 220μs yet more than 80μs.

Let us assume it to he 100μs. That means a pulse width of $2b = 100\mu$s. (See Fig. 3.)

If

$$\frac{a}{m_1} = a_0 = \text{acceleration,}$$

then

$$v = \frac{ab}{m_1} = a_0 b$$

For a 2000-ft. pound shock the table speed v is approximately 7 ft./sec. Substituting, we find for a_0 or the acceleration

$$7 = a_0 \times .00005$$

$$\text{or } a_0 = 140.000 \text{ ft./sec.}^2$$

$$a_0 = 4400 \text{ ``g''s}$$

This is about the order of magnitude which the accelerometers have recorded.

The important conclusion we draw from this is that the acceleration and its time interval combine to produce a velocity of the base which is a complete criterion of the severity of the shock administered.

In the example just cited the weight of the table is approximately 800 lbs., and the force 4400 x 800 = 3,250,000 lbs. The result is, then, that a triangular impulse of 3,520,000 lbs. magnitude and a duration of 100μs operating on a table of 800 lbs., imparts to that table a velocity of 7 ft./sec.

PART II

Analysis of the Response

In Part I the origin of the motion of the base has been treated. This motion of the base can now be represented by a pulse or a displacement as a function of time. To distinguish the displacement-time function from the force-time function, we have already suggested the name *Whip*. Obviously some of the pulse shapes which were used to represent impulses are not suitable as whips. For instance, the square pulse as whip could not exist, since this would suppose an infinite velocity.

The triangular whip is observed in the medium-high-impact shock machine. The sine whip may be taken to represent approximately the output of the light-high-impact machine.

The shifted cosine whip is sometimes used in the motion of cams of automatic equipment.

The problem of shock response is now reduced to the behavior of a mass and spring system when the base motion is represented by a whip.

Triangular whip (Fig. 3b). The Laplace transform of this pulse is

$$F(s) = \frac{a}{b}\left(\frac{1 - \epsilon^{-bs}}{s}\right)^2.$$

The differential equation for a simple harmonic system is

$$m\ddot{x} + kx = 0 \tag{1}$$

or

$$\frac{\ddot{x}}{\omega^2} + x = 0. \tag{2}$$

If we let the whip operate on this system, then

$$\frac{\ddot{x}}{\omega^2} + x = f(t) = x_1(t) \tag{3}$$

in which $x_1(t)$ represents the displacement of the whip as a function of time.

$$\text{Let } \mathfrak{L}[x(t)] = X(s)$$

and

$$\mathfrak{L}[x_1(t)] = \mathfrak{L}[f(t)] = F(s) = X_1(s)$$

then

$$\mathfrak{L}[\ddot{x}] = s^2X(s) - sf(0) - f^1(0)$$

By definition the initial conditions are zero, so that

$$\mathfrak{L}[\ddot{x}] = s^2X(s) \tag{4}$$

The Laplace transform of equation (3) is then

$$\frac{s^2}{\omega^2}X(s) + X(s) = \mathfrak{L}[f(t)] = X_1(s)$$

or

$$\left(\frac{s^2 + \omega^2}{\omega^2}\right)X(s) = X_1(s). \tag{5}$$

Now

$$X_1(s) = \frac{a}{b}\left(\frac{1 - \epsilon^{-bs}}{s}\right)^2.$$

Substituting and rearranging,

$$X(s) = \frac{\omega^2}{s^2 + \omega^2}\,\frac{a}{b}\left(\frac{1 - \epsilon^{-bs}}{s}\right)^2. \tag{6}$$

This is the transform equation. To find x we use the inverse Laplace transform and the solution of (7) is

$$x = \frac{a\omega^2}{b}\left[\left(\frac{t}{\omega^2} - \frac{\sin \omega t}{\omega^3}\right) - 2\left(\frac{(t-b)}{\omega^2} - \frac{\sin \omega(t-b)}{\omega^3}\right) u(t-b) + \left(\frac{t-2b}{\omega^2} - \frac{\sin \omega(t-2b)}{\omega^3}\right) u(t-2b)\right]. \tag{7}$$

The expression $u(t - b)$ simply means that the term to which it is attached is zero for all values of $t < b$.

Let us now consider what this solution consists of.

There are apparently three terms which take effect at successive intervals. The initial whip can be considered to consist of three different displacements starting at successive times 0, b and $2b$. With the displacement of the base there is a corresponding displacement of the mass m. After the time b the second term or displacement takes hold and an associated displacement of mass m except that the initial conditions are the end conditions of the first displacement. After the time $2b$ the third displacement enters and the final result is the displacement-time pulse or whip. To make the problem somewhat simpler we introduce the following modifications:

1°. Because the motion is a simple harmonic of known frequency after the whip has passed we will only consider the maximum amplitude.

2°. Only the displacement-time function of the mass m during the pulse interval will be examined.

3°. The dimensional magnitudes of the motion of mass m will be expressed as ratios of those of the pulse.

If a is the maximum amplitude of the whip, and $T_0 = 2b$ its time interval (usually expressed in milliseconds), then we define

$\dfrac{x}{a} = \delta$ *Amplitude ratio of pulse displacement and response of mass m during pulse interval only.*

$\dfrac{T_0}{T} = \dfrac{2b}{T} = \dfrac{2b}{2\pi/\omega} \dfrac{\omega b}{\pi} = \varphi$ *Natural frequency of mass m expressed as a ratio of the pulse length.*

$\tau = \dfrac{t}{2b}$ *Elapsed time expressed as a ratio of the pulse length.*

$\Delta = \dfrac{x_{\max}}{a}$ *Ratio of maximum amplitude to pulse displacement after pulse interval.*

Substituting these values in equation (7) and rearranging we obtain

$$\delta = 2\tau - \frac{\sin 2\varphi\pi\tau}{\pi\varphi} - 2\left((2\tau - 1) - \frac{\sin \pi\varphi(2\tau - 1)}{\pi\varphi}\right) u(2\tau - 1) + \left(2(\tau - 1) - \frac{\sin 2\pi\varphi(\tau - 1)}{\pi\varphi}\right) u(\tau - 1) \tag{8}$$

This looks somewhat complicated, but we can simplify by omitting the last term, because we are only considering values of δ during the pulse interval.

$$\therefore\ \delta = 2\tau - \frac{\sin 2\varphi\pi\tau}{\pi\varphi} - 2\left((2\tau - 1) - \frac{\sin \pi\varphi(2\tau - 1)}{\pi\varphi}\right)u(2\tau - 1) \quad (9)$$

A plot of this equation for various values of φ is shown in Fig. 4. It is seen that δ becomes a maximum when φ is approx. .9 and τ is then .75. The displacement is approximately 1.5 times the peak displacement of the whip.

After the whip has passed, or when $\tau > 1$, the transient has disappeared and a steady-state condition exists. Since the system under consideration is a simple harmonic system, the steady state is a harmonic motion of frequency ω, with an amplitude to be obtained from equation (8). Indicating the dimensionless values of the amplitude by δ_a when $\tau > 1$, equation 8 may be written

$$\begin{aligned}\delta_a = 2\tau - \frac{\sin 2\varphi\pi\tau}{\pi\varphi} - 2\left((2\tau - 1) - \frac{\sin \varphi(2\tau - 1)}{\pi\varphi}\right)& \\ + \left(2(\tau - 1) - \frac{\sin 2\pi\varphi(\tau - 1)}{\pi\varphi}\right) \qquad \tau > 1&\end{aligned} \quad (10)$$

After developing (10) and rearranging we obtain

$$\delta_a = \frac{2(1 - \cos \pi\varphi)}{\pi\varphi} \sin \pi\varphi(2\tau - 1). \quad (11)$$

The maximum amplitude is

$$\Delta = \frac{2(1 - \cos \pi\varphi)}{\pi\varphi} \cdot \quad (12)$$

A plot of equation (12) is shown in Fig. 5. Before considering the action of this whip in terms of what it does to the system, we shall take a brief look at the analysis of the two other whips; viz., the sine whip and shifted cosine whip (see Fig. 3).

Sine Whip

We have again equation (3).

$$\frac{\ddot{x}}{\omega^2} + x = f(t) = x_1(t)$$

and equation (5)

$$\left(\frac{s^2 + \omega^2}{\omega^2}\right) X(s) = F(s)$$

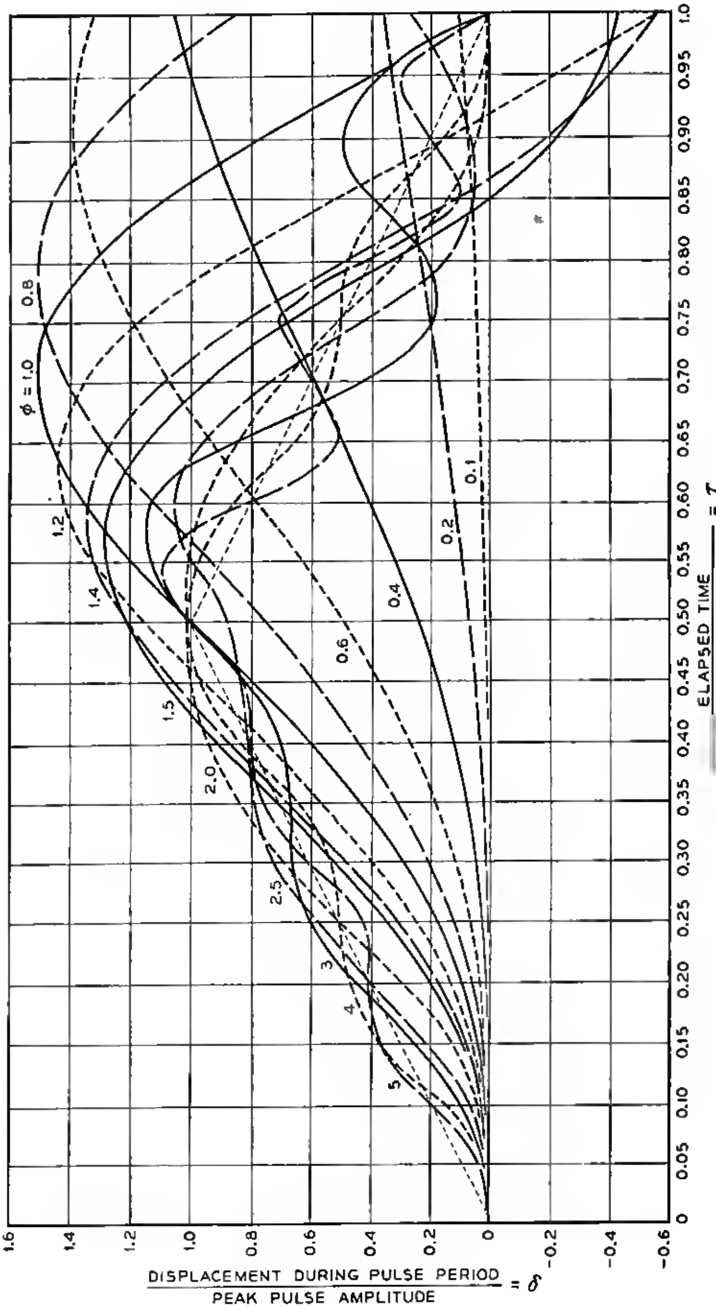

Fig. 4—Transient time displacement curves for various values of triangular whip.

From 1.04

$$F(s) = \frac{a\omega_0}{s^2 + \omega_0^2}\,(1 + \epsilon^{-\pi s/\omega_0}). \tag{13}$$

Substituting (13) in (5) we obtain

$$X(s) = \frac{a\omega^2\omega_0}{(s^2 + \omega^2)(s^2 + \omega_0^2)}\,(1 + \epsilon^{-\pi s/\omega_0}). \tag{14}$$

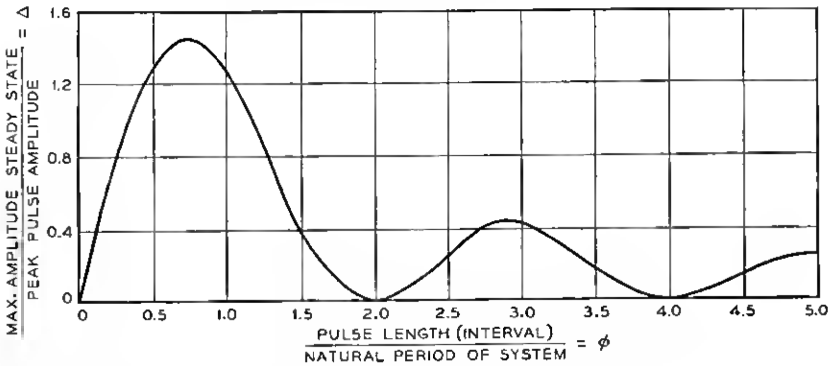

Fig. 5—Maximum amplitude as a function of frequency ratio-steady state triangular whip.

The inverse transform gives us

$$x = \frac{a\omega^2\omega_0}{\omega^2 - \omega_0^2}\left[\left(\frac{1}{\omega_0}\sin\omega_0 t - \frac{1}{\omega}\sin\omega t\right) + \left(\frac{1}{\omega_0}\sin\omega_0\left(t - \frac{\pi}{\omega_0}\right) - \frac{1}{\omega}\sin\omega\left(t - \frac{\pi}{\omega_0}\right)\right) u\left(t - \frac{\pi}{\omega_0}\right)\right] \tag{15}$$

and dividing this into two parts again, the transient and the steady state, we find for the transient,

$$x = \frac{a\omega^2\omega_0}{\omega^2 - \omega_0^2}\left[\frac{1}{\omega_0}\sin\omega_0 t - \frac{1}{\omega}\sin\omega t\right]$$

and substituting the dimensionless quantities

$$\delta = \frac{x}{a}, \qquad \frac{\pi/\omega_0}{T} = \varphi$$

and

$$\tau = \frac{t}{\pi/\omega_0}$$

we find

$$\delta = \frac{4\varphi^2}{4\varphi^2 - 1}\left(\sin\pi\tau - \frac{1}{2\varphi}\sin 2\pi\varphi\tau\right). \tag{16}$$

A plot of this equation as a family of curves for various values of φ is shown in Fig. 6. It is noted that, in general, this group of curves resembles those of Fig. 4 of the triangular whip.

The steady state is

$$x = \frac{a\omega^2\omega_0}{\omega^2 - \omega_0^2}\left[\frac{1}{\omega_0}\sin\omega_0 t - \frac{1}{\omega}\sin\omega t + \frac{1}{\omega_0}\sin\omega_0\left(t - \frac{\pi}{\omega_0}\right) - \frac{1}{\omega}\sin\omega\left(t - \frac{\pi}{\omega_0}\right)\right] \tag{17}$$

and, in dimensionless quantities or expressed as a ratio of the pulse dimensions, we obtain

$$\delta_a = \frac{4\varphi}{1 - 4\varphi^2}\cos\pi\varphi\sin 2\pi\varphi(\tau - 1). \tag{18}$$

From (18) it follows that the maximum amplitude of the steady state is

$$\Delta = \frac{4\varphi\cos\pi\varphi}{1 - 4\varphi^2}. \tag{19}$$

A plot of this curve is shown in Fig. 7.

Shifted Cosine Whip.

The shifted cosine whip produces results of a similar nature. We have seen that the transform equation for this whip is (1.05)

$$F(s) = \frac{a\omega_0^2}{2s(s^2 + \omega_0^2)}\left(1 - \epsilon^{-2\pi s/\omega_0}\right). \tag{20}$$

Using equations (3) and (5) and transferring to dimensionless quantities, in which

$$\frac{x}{a} = \delta, \quad \frac{2\pi/\omega_0}{T} = \varphi = \frac{\omega}{\omega_0}, \qquad \tau = \frac{t}{2\pi/\omega_0} = \frac{\omega_0 t}{2\pi}$$

we obtain

$$\delta = \frac{1}{2(\varphi^2 - 1)}\Big(\cos 2\pi\varphi\tau - \varphi^2\cos 2\pi\tau - \cos 2\pi\varphi(\tau - 1)u(\tau - 1) + \varphi^2\cos 2\pi(\tau - 1)u(\tau - 1)\Big). \tag{21}$$

Since we are interested only in the transient displacement, (21) becomes

$$\delta = \frac{(1 - \cos 2\pi\varphi\tau) - \varphi^2(1 - \cos 2\pi\tau)}{2(1 - \varphi^2)} \tag{22}$$

A family of curves showing δ for various values of φ is shown in Fig. 8.

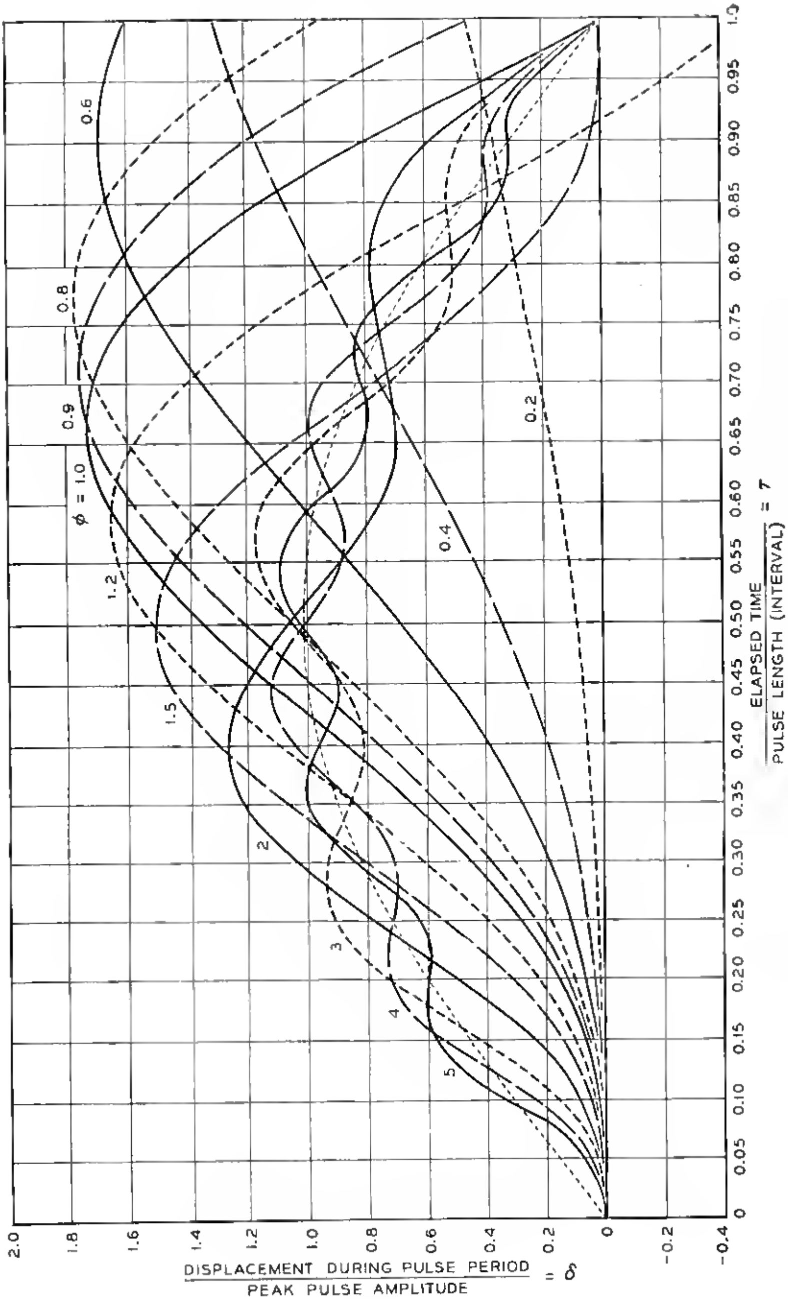

Fig. 6—Transient time displacement curves for various values of φ sine whip.

The steady state after the transient is (from Eq. 21)

$$\delta_a = \frac{1}{2(\varphi^2 - 1)} \Big(\cos 2\pi\varphi\tau - \varphi^2 \cos 2\pi\tau - \cos 2\pi\varphi(\tau - 1) + \varphi^2 \cos 2\pi(\tau - 1) \Big)$$

which reduces to

$$\delta_a = \frac{\sin \pi\varphi}{1 - \varphi^2} \sin (2\pi\varphi\tau - \pi\varphi). \tag{23}$$

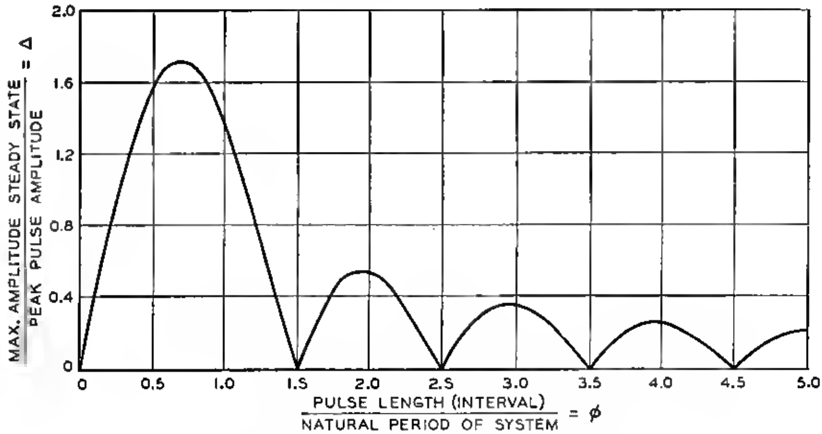

Fig. 7—Maximum amplitude as a function of frequency ration-steady state sine whip.

The maximum amplitude is

$$\Delta = \frac{\sin \pi\varphi}{1 - \varphi^2} \tag{24}$$

a plot of which is shown in Fig. 9.

Practical Considerations

Let us now consider the action of these various whips in terms of what they do to the system. The designer of shockmounts is primarily interested in the displacement across the mount or the relative displacement of base and mass.

In Fig. 10 the relative transient displacements for four systems are shown when subjected to a triangular whip. The natural frequencies are .4, 1.0, 1.5, and 2 times the frequency of the whip. From this it appears that the maximum relative displacement is approximately equal to the maximum

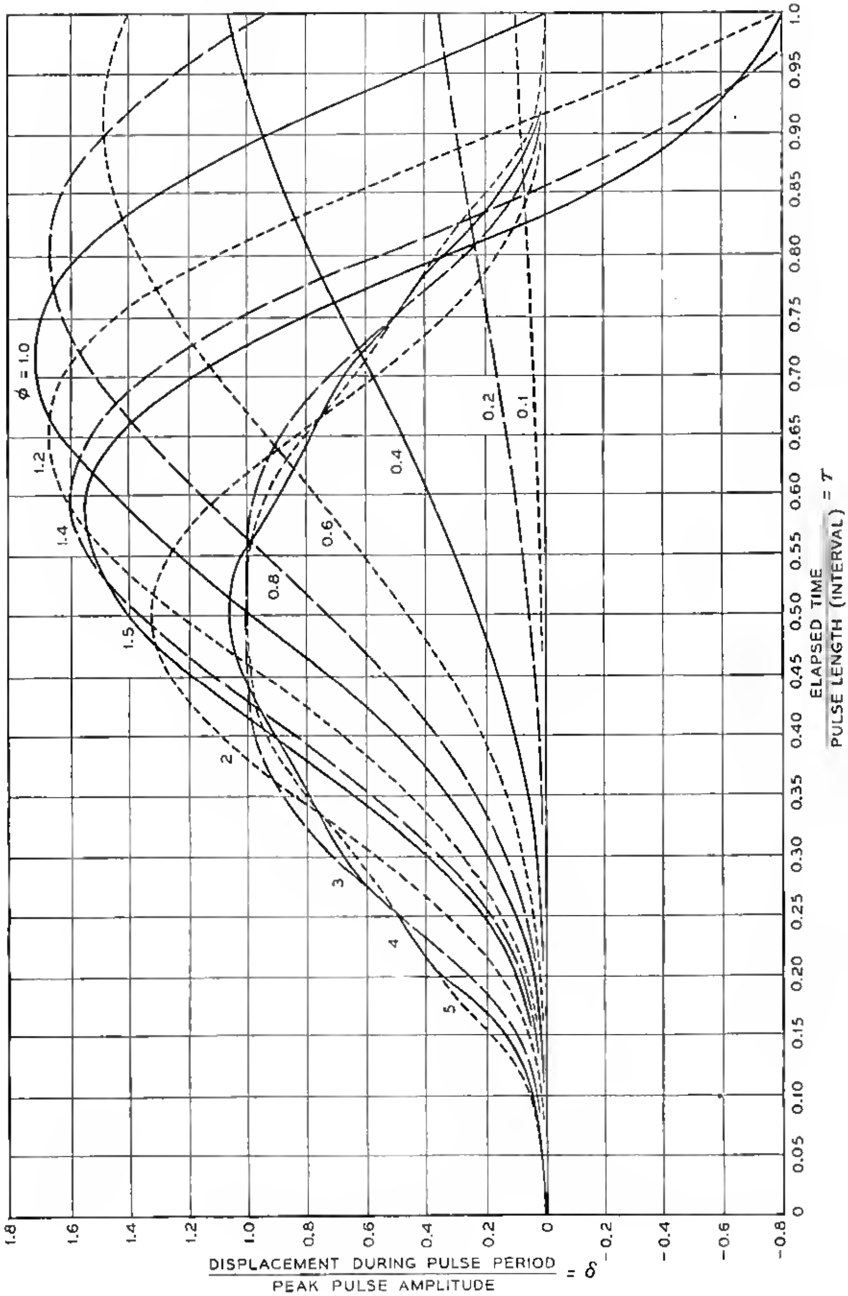

Fig. 8—Transient time displacement curves for various values of φ shifted cosine whip.

whip displacement. It is also observed that the large relative displacements occur when the frequency of the system is smaller than the pulse frequency.

After the transient has passed, the relative steady-state displacement, which is of course equal to the absolute, obtains large values too.

From Fig. 5 we note that a maximum of 1.5 is reached for the triangular whip and up to 1.7 times for the sine whip (see Fig. 7) at a frequency of approximately $\frac{3}{4}$ of the whip. Apparently even larger displacements across the mount occur after the transient has disappeared.

This is illustrated in Fig. 11 for the same systems as in Fig. 10.

As φ increases, which means if the frequency of the system increases with respect to the pulse frequency, the displacements across the mounts diminish,

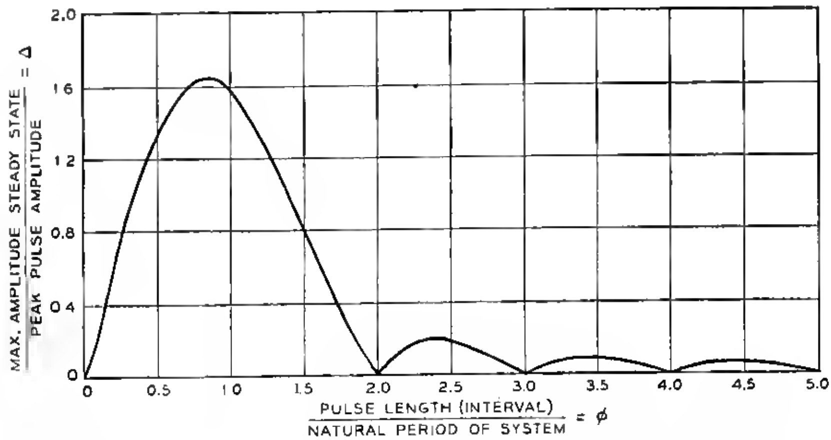

Fig. 9—Maximum amplitude as a function of frequency ratio-steady state shifted cosine whip.

while on the other hand the acceleration increases as will be shown later (see equation 39).

From this it seems advantageous to select a natural period of the system at least twice that of the pulse frequency.

The relative displacements are limited by practical considerations, such as available space between cabinets and bulk head, cable connections, personnel safety and others.

In the design of Bell Telephone Laboratories radar equipment, the relative displacement has been held to one-half inch, and the natural frequency in the neighborhood of 35 to 40 cycles per second or a period of 25 to 30 m.s.

The average of the heaviest shock administered to this type of equipment has a peak amplitude of 1.5 inch and a time interval of approximately 60 m.s.

From Fig. 5, we find that under these conditions a maximum relative dis-

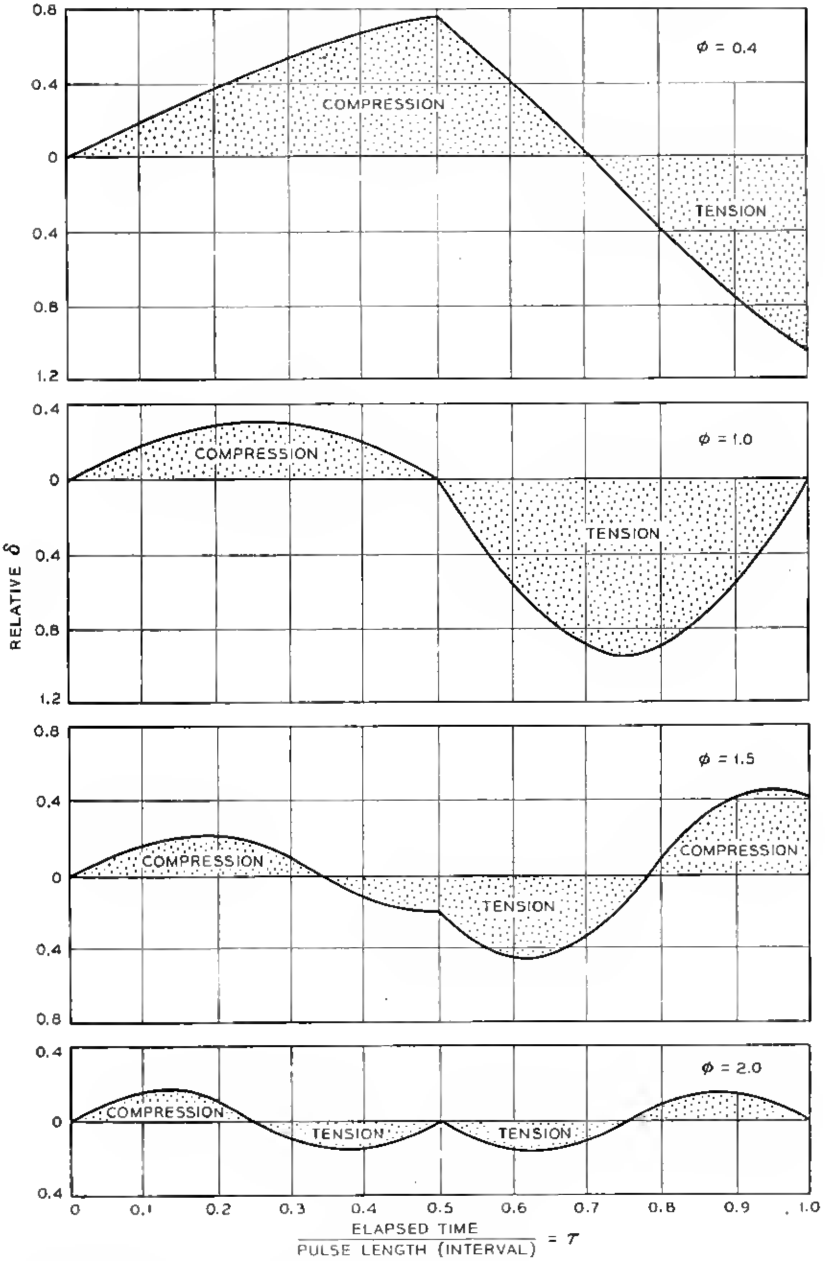

Fig 10—Transient time displacement curves across the mount for various values of φ triangular whip.

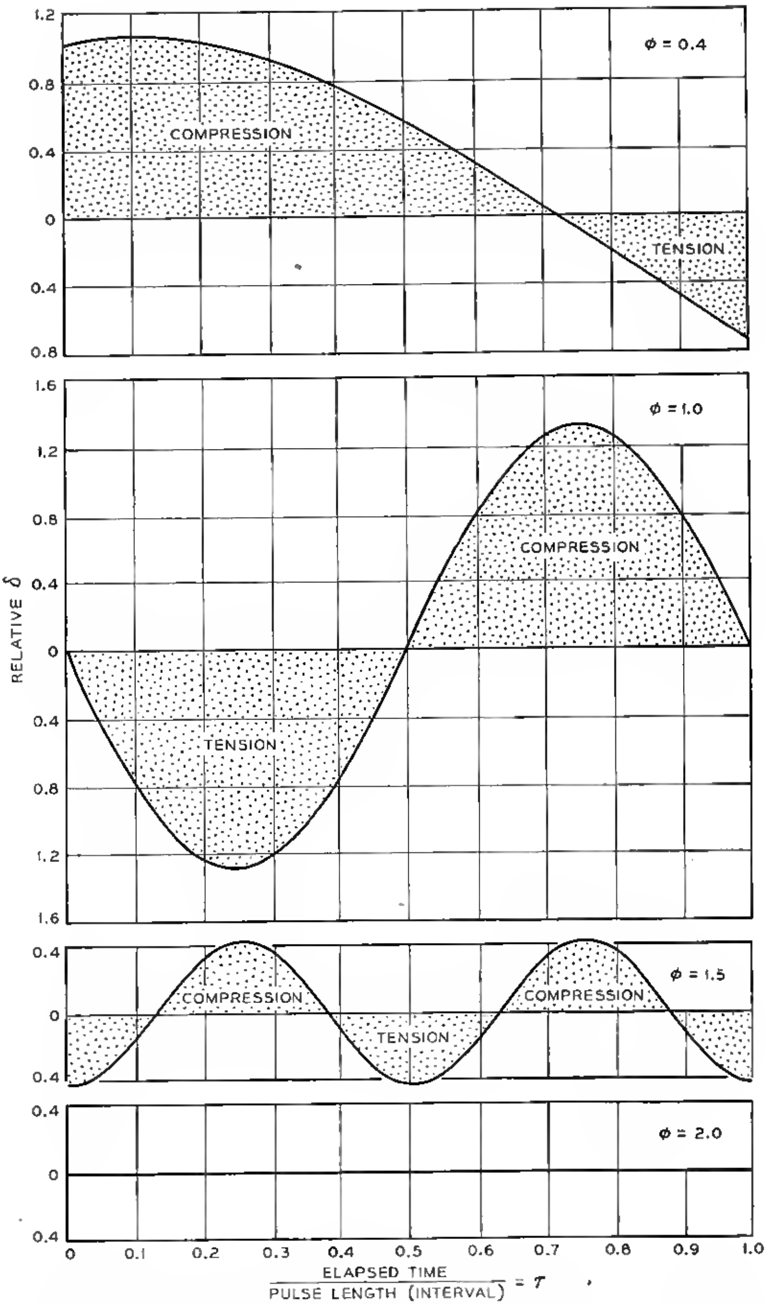

Fig. 11—Steady state time displacement curves across the mount. Triangular whip.

placement of .42 times the peak pulse amplitude or approximately $\frac{5}{8}$ inch may be expected.

Taking into consideration that the shock mount has been designed with a certain amount of damping, it is thus possible to hold the relative displacement within the boundaries of its shock-absorbing capacity.

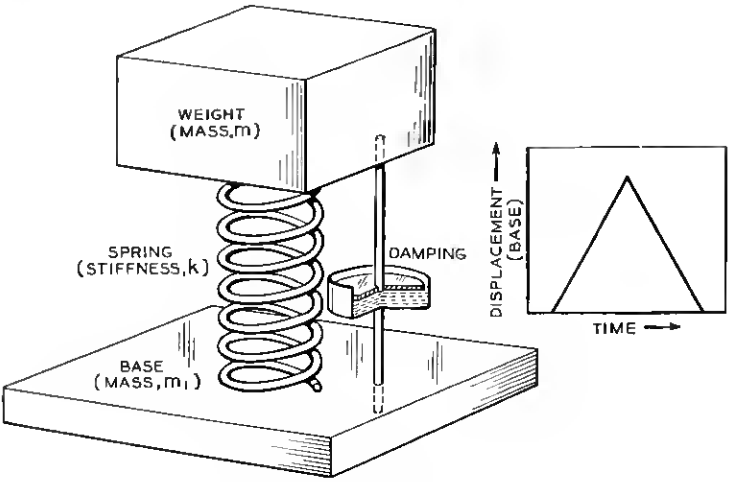

Fig. 12—System with damping.

Viscous Damping

The fundamental differential equation for a system with damping is (see Fig. 12).

$$\ddot{x} + 2\ell\dot{x} + \omega^2 x = 0 \tag{25}$$

If we let a whip operate on this system we obtain

$$\ddot{x} + 2\ell\dot{x} + \omega^2 x = \omega^2 x_1(t) \tag{26}$$

However, the sudden displacement of the base also produces an acceleration of the mass proportional to the velocity. If $x_1(t)$ is the displacement then $\dot{x}_1(t)$ may be represented to be the velocity and $2\ell\dot{x}_1(t)$ the acceleration. We have, then, for the completed equation

$$\ddot{x} + 2\ell\dot{x} + \omega^2 x = \omega^2 x_1(t) + 2\ell\dot{x}_1(t) \tag{27}$$

In the Laplacian terminology, if

$$x_1(t) = F(s)$$

then

$$\dot{x}_1(t) = sF(s) \quad \text{(initial value being zero)}$$

The Laplace transform of equation (27) is

$$X(s) = \frac{\omega^2 + 2\ell s}{s^2 + 2\ell s + \omega^2} F(s). \tag{28}$$

The solution of (28) is made easier if it is written in the form

$$X(s) = \frac{\omega^2 + 2\alpha s}{(s + \alpha)^2 + \beta^2} F(s) \tag{29}$$

in which

$$\alpha = \ell \qquad \text{and} \qquad \alpha^2 + \beta^2 = \omega^2.$$

Subjecting this system to a triangular whip, of which the Laplace transform is

$$F(s) = \frac{a}{b}\left(\frac{1 - \epsilon^{-bs}}{s}\right)^2$$

we have

$$X(s) = \frac{\omega^2 + 2\alpha s}{(s + \alpha)^2 + \beta^2} \frac{a}{b}\left(\frac{1 - \epsilon^{-bs}}{s}\right)^2 \tag{30}$$

the solution of which involves two transform pairs. The inverse transform gives us a solution of the transient as well as the steady state. It has been mentioned before that the steady state produces the maximum displacements across the mount; therefore it will be considered in more detail. We find that the steady state solution is

$$x_a(t) = \frac{a}{b} \frac{\epsilon^{-\alpha t}}{\beta}\left(-\sin \beta t + 2\epsilon^{\alpha b} \sin \beta(t - b) - \epsilon^{2\alpha b} \sin \beta(t - 2b)\right) \tag{31}$$

Which simplifies to

$$x_a(t) = \frac{\epsilon^{-\alpha t}}{\beta} \frac{a}{b} \sqrt{(A^2 + B^2)} \sin (\beta t - \theta) \tag{32}$$

Using dimensionless quantities

$$\eta = \frac{\ell}{\omega} = \frac{\alpha}{\omega} \qquad \text{and} \qquad \frac{x_a}{a} = \delta_a$$

and the substitution

$$\sqrt{1 - \eta^2} = \gamma,$$

we find that

$$b\beta = b\omega\gamma = \pi\varphi\gamma$$

Equation (32) may now be expressed as

$$\delta_a = \frac{\epsilon^{-\alpha t}}{\pi\varphi} \sqrt{(A^2 + B^2)} \sin(\omega\gamma t - \theta) \quad \alpha t > 2\eta\pi\varphi \tag{33}$$

in which

$$\tan\theta = \frac{B}{A}$$

and

$$A = -1 + 2\epsilon^{\eta\pi\varphi} \cos\pi\varphi\gamma - \epsilon^{2\eta\pi\varphi} \cos 2\pi\varphi\gamma \tag{34}$$

$$B = 2\epsilon^{\eta\pi\varphi} \sin\pi\varphi\gamma - \epsilon^{2\eta\pi\varphi} \sin 2\pi\varphi\gamma \tag{35}$$

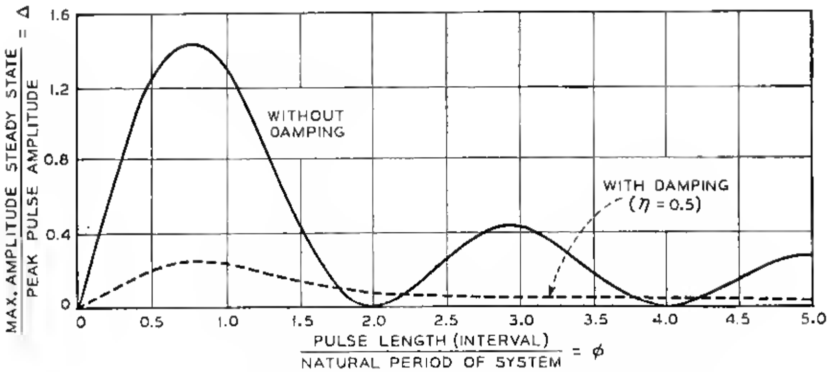

Fig. 13—Effect of damping on steady state amplitude for triangular whip.

From equation (33) we obtain the maximum displacement

$$\Delta = \frac{\epsilon^{-\alpha t}}{\pi\varphi} \sqrt{A^2 + B^2} \qquad \alpha t > 2\eta\pi\varphi \tag{36}$$

in which

$$\alpha t = \frac{\eta}{\gamma}\left(\tan^{-1}\frac{\gamma}{\eta} + \tan^{-1}\frac{B}{A}\right) \qquad \alpha t > 2\eta\pi\varphi$$

In Fig. 13 a plot of equation (36) is shown for $\eta = .5$. This indicates that the peak value of Δ is .24 as compared to 1.48 when no damping is present.

Accelerations

The transient accelerations of the mass m during the whip action and the subsequent steady state may be found by examining the acceleration during the first part of a triangular whip. Designating the velocity of displacement

of the whip by v the expression for x may be formed from equation (7) by setting $t < b$.

$$x = \frac{a\omega^2}{b}\left(\frac{t}{\omega^2} - \frac{\sin \omega t}{\omega^3}\right), \qquad t < b \qquad (37)$$

and, since $\frac{a}{b} = v$

$$x = vt - \frac{v}{\omega} \sin \omega t \qquad (38)$$

whence

$$\ddot{x} = v\omega \sin \omega t$$

and the maximum acceleration is

$$A_0 = v\omega \qquad (39)$$

Let

$$\frac{A_0}{v\omega} = \lambda_0 \text{ then } \lambda_0 = 1$$

By proceeding in a similar manner with the next step of the whip the value of the acceleration ratio will be found to be

$$\lambda_0 = 3$$

and for the completed whip or steady state

$$\lambda_0 = 4$$

The expression $A_0 = \lambda_0 v\omega$ is an important factor in shock considerations. Thus we have a simple relation for the final maximum amplitude of the periodic acceleration of the mass m when subjected to a triangular whip; viz., it is four times the product of whip velocity and natural frequency of the system. The constant λ_0 depends upon the configuration of the whip; the velocity v indicates the intensity of the whip; while ω expresses the kind of response the system is capable of.

It is of interest to note that this maximum periodic value of A_0 will be produced only if the ratio of pulse frequency and natural frequency is of the correct value. It is difficult to produce shocks on existing equipment of exactly the same characteristics within narrow limits as to time duration and therefore it must expected that a considerable variation in damage may occur even though similar shocks are administered to identical test objects. For the same reason a shock of lower intensity may produce

more damage than a higher one because impulse amplitude as well as duration change at the same time.

Although damping is a highly desirable feature in a shock mount, the damping device may cause a certain amount of coupling between the mass and the base, and if too much damping is provided the transient acceleration of the mass may become excessive.

The analysis of this problem by means of the Laplace transforms is not difficult, for we can use results previously obtained. The transform equation for a system with damping, subjected to a whip, is

$$X(s) = \frac{\omega^2 + 2\alpha s}{(s + \alpha)^2 + \beta^2} F(s) \tag{40}$$

in which $F(s)$ represents the transform of the disturbance or excitation. Since we are interested in the effect of the damping or η upon the response, only the first part of the triangular whip will be considered.

In this case

$$x_1(t) = vt$$

and

$$\mathfrak{L}[x_1(t)] = F(s) = \frac{v}{s^2}. \tag{41}$$

Substituting (41) in (40)

$$X(s) = \frac{v(\omega^2 + 2\alpha s)}{s^2[(s + \alpha)^2 + \beta^2]} = F_1(s) \tag{42}$$

If $X(s)$ is the transform of $x(t)$, a displacement, then the acceleration is $\ddot{x}(t)$ or $g(t)$, ($\ddot{x}(t) = g(t)$ by definition) and

$$\mathfrak{L}[\ddot{x}(t)] = \mathfrak{L}[g(t)] = s^2Xs$$

Substitution in (42) gives

$$s^2 X(s) = 2v\alpha \frac{s + \dfrac{\omega^2}{2\alpha}}{(s + \alpha)^2 + \beta^2} \tag{43}$$

Now $\mathfrak{L}^{-1}[s^2X(s)] = g(t)$
so that

$$g(t) = \frac{2v\alpha}{\beta}\left[\left(\frac{\omega^2}{2\alpha} - \alpha\right)^2 + \beta^2\right]^{\frac{1}{2}} \epsilon^{-\alpha t} \sin(\beta t + \psi) \tag{44}$$

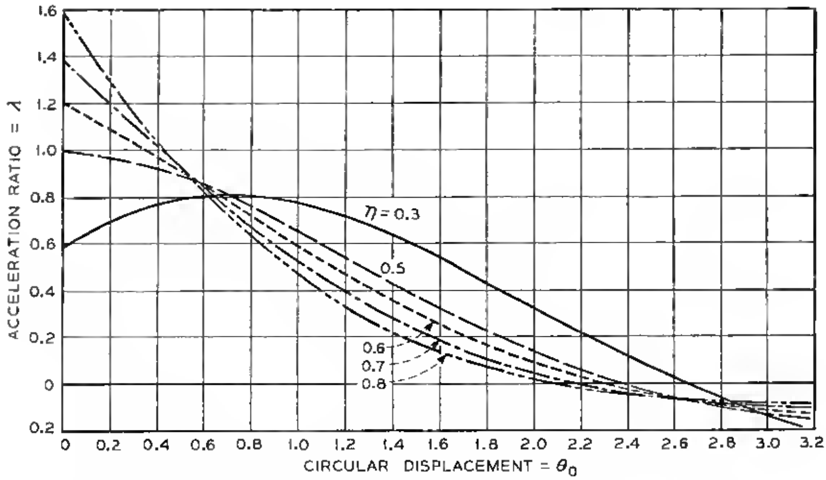

Fig. 14—Transient acceleration during initial part of triangular whip.

Letting

$$\frac{g(t)}{v\omega} = \lambda, \qquad \frac{\alpha}{\omega} = \frac{\ell}{\omega} = \eta, \qquad \beta = \omega\sqrt{1 - \eta^2} = \omega\gamma$$

and

$$\omega t = \theta_0$$

Substituting

$$\lambda = \frac{\epsilon^{-\eta\theta_0}}{\gamma} \sin(\lambda\theta_0 + \psi) \tag{45}$$

in which

$$\tan\psi = \frac{2\eta\gamma}{1 - 2\eta^2}$$

Figure 14 is a plot of λ against θ_0 for various values of η. It is noted that for $\eta = .5$ of critical damping the initial acceleration is equal to the undamped, λ being one.

The data presented here are also applicable to long duration pulses, because the final results have been given in dimensionless quantities, the units of measurement being those of the pulse.

Symbols Used

Mass .... m
Mass of base.. m_1
Spring stiffness.. k

Displacement mass . . . . x, $x(t)$
Displacement base . . . . x_1, $x_1(t)$
Force on base . . . . F, $F(t)$
Velocity of base v
Acceleration of base . . . . a_0 , $\ddot{x}_1$
Acceleration of mass m . . . . $\ddot{x}$, $\ddot{x}(t)$, $g(t)$
Maximum acceleration of the mass m . . . . A_0
Maximum acceleration ratio . . . . $\lambda_0 = \dfrac{A_0}{v\omega}$
Acceleration ratio . . . . $\lambda = \dfrac{g(t)}{v\omega}$
Natural frequency of mass m (circular) . . . . ω, β_0
Circular (angular) displacement . . . . $\omega t = \theta_0$
Frequency of sinusiode of which pulse consists (not pulse frequency) . . . . ω_0
Peak pulse displacement . . . . a

$$\left.\begin{array}{ll} \text{Pulse period (triangular)} \ldots\ldots & 2b \\ \text{`` `` (sine pulse)} \ldots\ldots & \dfrac{\pi}{\omega_0} \\ \text{`` `` (shifted cosine pulse)} \ldots\ldots & \dfrac{2\pi}{\omega_0} \end{array}\right\} T_0$$

Period of mass m . . . . T

$$\frac{\text{displacement during pulse period}}{\text{peak pulse amplitude}} = \ldots\ldots \frac{x}{a} = \delta$$

$$\frac{\text{amplitude steady state}}{\text{peak pulse amplitude}} = \ldots\ldots \delta_a$$

$$\frac{\text{max. amplitude steady state}}{\text{peak pulse amplitude}} = \ldots\ldots \Delta$$

$$\frac{\text{elapsed time}}{\text{pulse length (interval)}} = \ldots\ldots \frac{t}{2b}, \frac{t/\pi}{\omega_0}, \frac{t/2\pi}{\omega_0}, \frac{t}{T_0} = \tau$$

$$\frac{\text{pulse length (interval)}}{\text{natural period of system}} \ldots\ldots \frac{T_0}{T} = \varphi$$

Damping coefficient . . . . ℓ, α
Critical damping ratio . . . . $\dfrac{\ell}{\omega} = \eta$
Transform of $x(t)$ = ... $X(s)$
" " $x_1(t)$ = ... $X_1(s)$
" " $F(t)$ = ... $F_0(s)$
" " $f(t)$ = ... $F(s)$

$f(t)$ represents any function of t, without reference to its dimensional magnitude.